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Free field realization of $SL(2)$ correlators for admissible representations, and hamiltonian reduction for correlators

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A presentation is given of the free field realization relevant to $SL(2)$ WZW theories with a Hilbert space based on admissible representations. It is known that this implies the presence of two screening charges, one involving a fractional power of a free field. We develop the use of fractional calculus for treating in general such cases. We derive explicit integral representations of N-point conformal blocks. We show that they satisfy the Knizhnik-Zamolodchikov equations and we prove how they are related to minimal conformal blocks via a formulation of hamiltonian reduction advocated by Furlan, Ganchev, Paunov and Petkova.

1. INTRODUCTION

In this talk I shall describe how to obtain conformal blocks for $SL(2)$ WZW theories in the case of non-integrable representations, in particular for admissible representations [1,2]. There are several reasons why this would be of interest. First, a rather more complicated structure than is present for integrable representations and in minimal models reveals itself, and it is interesting to study in its own right. Second, our own principal interest comes from refs.[3,4] in which it has been demonstrated in principle how 2-d quantum gravity coupled to minimal conformal matter may be described in terms of a topological G/G model, with $G = SL(2)$. The way minimal conformal matter arises is related to viewing these theories in terms of a hamiltonian reduction of an $SL(2)$ theory [5–7], and it is well known that this relation depends on admissible representations. Finally, obtaining conformal blocks

for non-integrable representations may be interesting in various other ways, in particular in relations to formulations of black hole string solutions as discussed by Bars [8].

There already exists a number of approaches towards the problem of conformal blocks for admissible representations in the literature [9–17,2,18–21]. However, our goal here [22] is to obtain a formulation based on the Wakimoto [23] free field realization, and despite several attempts a complete solution has so far been missing. The principal reason for this is related to the need for introducing a second screening charge in the case of admissible representations. This screening operator involves a fractional power of a free (antighost) field [7]. Also the formalism requires the introduction of several other fractional powers of free fields. It appears that this situation cannot be resolved by bosonization. Instead I shall describe how everything may be treated rather neatly by means of fractional calculus. The result is that we appear to have a straightforward free field formalism.

As an explicit verification that our formalism works we have managed to prove that the conformal blocks we obtain satisfy the Knizhnik-Zamolodchikov equations [24,25]. As an additional bonus we are able to provide a proof of an interesting suggestion by Furlan, Ganchev, Paunov and Petkova [11] for how conformal blocks of the $SL(2)$ WZW theory reduce to the conformal blocks of minimal models. Even though the relation is often of a singular nature, the result is very attractive.

When the level of the affine $\widehat{SL(2)}_k$ algebra is k, we define

$$k + 2 \equiv t = p/q \tag{1}$$

where p, q are positive coprime integers for admissible representations. For these there are degenerate representations whenever the spin is given by

$$
\begin{aligned}
2j_{r,s} + 1 &= r - st \\
r &= 1, ..., p - 1 \\
s &= 0, ..., q - 1
\end{aligned}
\tag{2}
$$

The relation to minimal conformal models is then as follows

$$
\begin{aligned}
h_{r,s+1} &= \frac{1}{t} j_{r,s}(j_{r,s} + 1) - j_{r,s} \\
&= \frac{1}{2}\alpha_{r,s+1}(\alpha_{r,s+1} - 2\alpha_0) \\
\alpha_{r,s+1} &= -j_{r,s}\sqrt{\frac{2}{t}} = \frac{1}{2}[(1-r)\alpha_+ - s\alpha_-] \\
\alpha_+ &= \sqrt{\frac{2}{t}} = -\frac{2}{\alpha_-} \\
2\alpha_0 &= \alpha_+ + \alpha_- \\
c &= 1 - 12\alpha_0^2 = 1 - \frac{6(p-q)^2}{pq}
\end{aligned}
\tag{3}
$$

where h is the conformal dimension and c is the central charge for the minimal model. So we see the need for understanding the case of non-integer levels and spins.

2. THE FREE FIELD REALIZATION

The Wakimoto free field realization is in terms of a scalar field, φ and a pair of bosonic dimension

$(1, 0)$ ghosts, (β, γ):

$$
\begin{aligned}
\varphi(z)\varphi(w) &\sim \log(z - w) \\
\beta(z)\gamma(w) &\sim \frac{1}{z - w} \\
J^+(z) &= \beta(z) \\
J^3(z) &= -\gamma\beta(z) - \sqrt{\frac{t}{2}}\partial\varphi(z) \\
J^-(z) &= -\gamma^2\beta(z) + k\partial\gamma(z) \\
&\quad - \sqrt{2t}\gamma\partial\varphi(z)
\end{aligned}
\tag{4}
$$

They satisfy

$$
\begin{aligned}
J^+(z)J^-(w) &\sim \frac{2}{z-w}J^3(w) + \frac{k}{(z-w)^2} \\
J^3(z)J^\pm(w) &\sim \pm\frac{1}{z-w}J^\pm(w) \\
J^3(z)J^3(w) &\sim \frac{k/2}{(z-w)^2} \\
c &= \frac{3k}{t}
\end{aligned}
\tag{5}
$$

Fateev and Zamolodchikov [26] introduced a very useful formalism for primary fields, which we shall adopt. In general there is a multiplet of primary fields $\phi_j^m(z)$. We combine these introducing an extra variable, x as follows

$$\phi_j(z, x) = \sum_m \phi_j^m(z) x^{j-m} \tag{6}$$

For integrable representations $2j$ is an integer and we simply get a polynomial in x. However, for fractional spins we have highest weight, lowest weight or continuous representations, and the x-dependence can be arbitrarily complicated. The new primary field satisfies the following OPE

$$
\begin{aligned}
J^a(z)\phi_j(w, x) &\sim \frac{1}{z-w}D_x^a\phi_j(w, x) \\
D_x^+ &= -x^2\partial_x + 2xj \\
D_x^3 &= -x\partial_x + j \\
D_x^- &= \partial_x
\end{aligned}
\tag{7}
$$

Correlators now have an additional projective invariance related to the x variable, thus the 3-point function satisfies

$$\langle\phi_{j_3}(z_3, x_3)\phi_{j_2}(z_2, x_2)\phi_{j_1}(z_1, x_1)\rangle$$

$$= C_{123} \frac{(x_2 - x_1)^{j_1+j_2-j_3}(x_2 - x_3)^{j_2+j_3-j_1}}{(z_2 - z_1)^{h_1+h_2-h_3}(z_2 - z_3)^{h_2+h_3-h_1}}$$

$$\times \frac{(x_1 - x_3)^{j_1+j_3-j_2}}{(z_1 - z_3)^{h_1+h_3-h_2}} \tag{8}$$

Notice that for $x_i = z_i$ this reduces to the 3-point function of the corresponding minimal model operators by virtue of eq.(3). The observation of [11] was that a similar situation seems to occur for any N-point conformal block. This we shall prove towards the end. One easily verifies that

$$\phi_j(z, x) = [1 + x\gamma(z)]^{2j} e^{-j\sqrt{\frac{2}{t}}\varphi(z)} \tag{9}$$

Finally there are the two screening charge currents [7]

$$\begin{aligned} S_1(z) &= \beta(z)e^{\sqrt{\frac{2}{t}}\varphi(z)} \\ S_{-t}(z) &= \beta(z)^{-t}e^{-t\sqrt{\frac{2}{t}}\varphi(z)} \end{aligned} \tag{10}$$

We see in these last equations the need for being able to treat fractional powers of free fields.

2.1. Fractional calculus

Our treatment of Wick contractions is based on the following identity, trivially valid for $-t$ a positive integer, but non-trivial for general t:

$$\begin{aligned} &\beta(z)^{-t}F(\gamma(w)) \\ &= \; : [\beta(z) + \frac{1}{z-w}\partial_{\gamma(w)}]^{-t} F(\gamma(w)) : \\ &= \sum_{n\in\mathbb{Z}} \binom{-t}{n} : \beta^n(z)(z-w)^{t+n} \\ &\quad \partial_{\gamma(w)}^{-t-n} F(\gamma(w)) : \end{aligned} \tag{11}$$

Examples of the use of fractional calculus [27] are the Riemann-Liouville operator

$$\partial^{-a}f(z) = \frac{1}{\Gamma(a)}\int_0^z (z-t)^{a-1}f(t)dt, \; a > 0 \tag{12}$$

and

$$\partial_x^a x^b = \frac{\Gamma(b+1)}{\Gamma(b-a+1)}x^{b-a} \tag{13}$$

In addition we shall need unconventional (asymptotic) expansions like

$$e^x = \partial_x^a e^x = \sum_{n\in\mathbb{Z}} \frac{1}{\Gamma(n-a+1)}x^{n-a} \tag{14}$$

for x an operator.

3. CONFORMAL BLOCKS

According to the above, we may treat the free field representation of an N-point conformal block in terms of the following integral of a free field correlator:

$$\langle j_N| \prod_{n=2}^{N-1} [1 + x_n\gamma(z_n)]^{2j_n} e^{-j_n\sqrt{\frac{2}{t}}\varphi(z_n)}$$

$$\times \oint \prod_{k=1}^{s} \frac{dv_k}{2\pi i}\beta^{-t}(v_k)e^{-t\sqrt{\frac{2}{t}}\varphi(v_k)}$$

$$\times \oint \prod_{l=1}^{r} \frac{dw_l}{2\pi i}\beta(w_l)e^{\sqrt{\frac{2}{t}}\varphi(w_l)}|j_1\rangle \tag{15}$$

Here r and s are the number of screening charges of the first and second kind respectively. I refer to [22] for a discussion of the precise choice of the bra and ket (see also below). The notation implies that the corresponding primary fields have been placed at $(z_1, x_1) = (0, 0)$ and $(z_N, x_N) = (\infty, \infty)$, thereby partly fixing the global $SL(2)$ invariances related to z and to x. We must now figure out how to do the contractions.

3.1. The three point function

In this case we have $j_1 + j_2 - j_3 = r - st$ with r, s being the number of screening operators. Due to projective invariance we know that the x dependence of the three point function will be of the form (cf. eq.(8))

$$\langle j_3|\{\phi_{j_2}(z, x)\}_{j_1}^{j_3}|j_1\rangle \propto x^{r-st} \tag{16}$$

Here $\{\phi_{j_2}(z, x)\}_{j_1}^{j_3}$ is the intertwining field defined by means of the screening charges [28]. This tells us that we should expand

$$[1 + x\gamma(z)]^{2j_2}$$

$$= \sum_{n\in\mathbb{Z}} \binom{2j_2}{n+r-st} [x\gamma(z)]^{n+r-st} \tag{17}$$

We also see that thus we are going to find the same net power of γ's as we have og β's. One trick now is to employ

$$(1 + x\gamma)^{2j}$$

$$= \Gamma(2j+1)\oint \frac{du}{2\pi i}\frac{1}{u}(\frac{D}{u})^{-2j}\exp(\frac{1+x\gamma}{u})$$

$$\beta^a(w)e^{\frac{1+x\gamma(z)}{u}}$$
$$= \ :(\beta(w)+\frac{1}{w-z}\partial_{\gamma(z)})^a \exp\left[\frac{1+x\gamma(z)}{u}\right]:$$
$$= \ :(\beta(w)+\frac{x/u}{w-z})^a D^a \exp(\frac{1+x\gamma(z)}{u}) : (18)$$

Here D represents differentiation wrt the argument of the exponential function. Now it is relatively straightforward to write down the integral representation of the three point function. The result is

$$
\begin{aligned}
W_3 \ = \ & \frac{\Gamma(2j_2+1)}{\Gamma(2j_2-r+st+1)} \int \prod_{i=1}^{r}\frac{dw_i}{2\pi i} \prod_{j=1}^{s}\frac{dv_j}{2\pi i} \\
& \prod_{i_1<i_2}(w_{i_1}-w_{i_2})^{2/t} \prod_{j_1<j_2}(v_{j_1}-v_{j_2})^{2t} \\
& \prod_{i,j}(w_i-v_j)^{-2} \\
& \prod_{i=1}^{r} w_i^{(1-r_1)/t+s_1}(1-w_i)^{(1-r_2)/t+s_2-1} \\
& \prod_{j=1}^{s} v_j^{r_1-1-s_1 t}(1-v_j)^{r_2-1-(s_2-1)t} \quad (19)
\end{aligned}
$$

This result [22] is a Dotsenko-Fateev integral [29], which may be analysed [28] to provide the following fusion rule $(2j_i+1=r_i-s_i t)$

$$
\begin{aligned}
1+|r_1-r_2| \leq \quad r_3 \quad \leq p-1-|r_1+r_2-p| \\
|s_1-s_2| \leq \quad s_3 \quad \leq q-1-|s_1+s_2-q+1|
\end{aligned}
$$
$$(20)$$

This agrees with results in refs.[19,20] referred to as their rule I. However these authors also provide a rule II:

$$
\begin{aligned}
1+|p-r_1-r_2| \leq \quad r_3 \quad \leq p-1-|r_1-r_2| \\
1+|q-s_1-s_2-1| \leq \quad s_3 \quad \leq q-2-|s_1-s_2|
\end{aligned}
$$
$$(21)$$

This rule appears not to follow from our three point function. However, very recently we have realized that (i) this rule is required in our 4-point functions and (ii) may formally be derived from our 3-point function by continuing to a negative number of screening charges. We intend to come back elsewhere with a more detailed discussion.

3.2. The general N-point block on the sphere

Using the techniques described it is possible to write down the general N-point function with $M=r+s$ screening charges as follows

$$
\begin{aligned}
W_N \ = \ & \int \prod_{i=1}^{M}\frac{dw_i}{2\pi i}W_N^{\varphi}W_N^{\beta\gamma} \\
W_N^{\varphi} \ = \ & \prod_{m<n}(z_m-z_n)^{2j_m j_n/t} \\
& \times \prod_{i=1}^{M}\prod_{m=1}^{N-1}(w_i-z_m)^{2k_i j_m/t} \\
& \times \prod_{i<j<M}(w_i-w_j)^{2k_i k_j/t} \\
W^{\beta\gamma} \ = \ & \int \prod_{m=2}^{N-1}\frac{du_m}{2\pi i}\Gamma(2j_m+1) \\
& \cdot \quad u^{2j_m-1}e^{1/u_m}\prod_{i=1}^{M}B(w_i)^{-k_i} \\
B(w_i)^{-k_i} \ = \ & (\sum_{l=1}^{N-1}\frac{x_l/u_l}{w_i-z_l})^{-k_i} \quad (22)
\end{aligned}
$$

where $z_1=x_1=0$. Here the powers k_i are -1 and t respectively for screening charges of the first and second kind. Notice that there are some simple rules for how to construct the $\beta\gamma$ part of the contractions. In particular

$$\beta(w_i)^{-k_i} \to B(w_i)^{-k_i} \quad (23)$$

The above integral representation for N-point blocks is our main result.

4. CHECKS ON THE RESULT

We have performed several consistency checks on the above result, some of which I mention here. So far we have constructed the conformal block as a correlator

$$\langle j_N|\phi_{N-1}(z_{N-1},x_{N-1})...\phi_{j_2}(z_2,x_2)|j_1\rangle \quad (24)$$

Here the chiral vertex operators, $\phi_l(z_l,x_l)$ are to be understood as screened intertwining fields. The precise choice of screening contours define which particular conformal block we are considering. This construction presupposes that the formalism is properly $SL(2)$ invariant both as far as

z and x variables go. However, we might also consider the construction based on

$$\langle 0|\phi_{j_N}(z_N, x_N)...\phi_{j_1}(z_1, x_1)|0\rangle \qquad (25)$$

In the limits

$$\begin{aligned} z_N, x_N &\rightarrow \infty \\ z_1, x_1 &\rightarrow 0 \end{aligned} \qquad (26)$$

the two should agree. This we have checked. In fact the choice of dual states and dual vacuum is not manifestly $SL(2)$ invariant [22,30]. We use bra and ket states with the following properties

$$\begin{aligned} \langle 0|0\rangle &= 1 \\ \langle 0|\gamma_0 &= 0 \\ \langle 0|\beta_0 &\neq 0 \\ \langle j| &= \langle 0|e^{j\sqrt{\frac{2}{t}}q_\varphi} \end{aligned} \qquad (27)$$

where q_φ is the position operator for the scalar field, φ. Hence we should also check that it is possible to prove that, nevertheless, the formalism is $SL(2)$ invariant. This we have done.

As a particular example of associativity (and other formal properties) we have checked in great detail that

$$[\beta^a(z)\gamma^a(w)][\beta^b(z)\gamma^b(w)] = \beta^{a+b}(z)\gamma^{a+b}(w) \quad (28)$$

Probably the most interesting and stringent test we have performed is to present a detailed proof that our correlators satisfy the Knizhnik-Zamolodchikov equations [24]. This means that our formalism constitutes a very powerful technique for generating solutions to these equations. The KZ equations may be written as

$$\{t\partial_{z_{m_0}} + 2\sum_{m\neq m_0} \frac{D^a_{x_{m_0}}D^a_{x_m}}{z_m - z_{m_0}}\}W_N = 0 \qquad (29)$$

One possible way of proving this equation is to consider the functions, $G^a(w)$ and $G(w)$ defined for a correlator of operators $\mathcal{O}$ by

$$\begin{aligned} G^a(w) &= \langle J^a(w)\mathcal{O}\rangle \\ G(w) &= \frac{1}{w - z_{m_0}}\{D^+_{x_{m_0}}G^-(w) \\ &+ 2D^3_{x_{m_0}}G^3(w) + D^-_{x_{m_0}}G^+(w)\} \end{aligned} \qquad (30)$$

For

$$\mathcal{O} = \phi_{j_N}(z_N, x_N)...\phi_{j_1}(z_1, x_1) \qquad (31)$$

one easily sees that $G(w)$ only has pole singularities at points $w = z_m$ and no pole at infinity:

$$G(w) \sim \mathcal{O}(1/w^2) \qquad (32)$$

Then the condition

$$\sum_m \text{Res}G|_{w=z_m} = 0 \qquad (33)$$

is the KZ equation. In our case we may build the function $G(w)$ as well. If our formalism based on fractional calculus is guaranteed to have all the correct associativity properties of the operator algebra, it is trivial that we should find the same result. However, that is what we want to check. Hence we build the function $G(w)$ using our rules. We observe that it has pole singularities at $w = z_m$, but in addition there are singularities also at $w = w_i$, the positions of the screening charge currents before they are integrated over. However, we may prove that these residues are total derivatives so that those contributions vanish. Finally the residues of $G(w)$ at $w = z_m$ turn out to be exactly the different terms in the KZ equations for our conformal block. This completes (a sketch of) the proof.

5. HAMILTONIAN REDUCTION

The N-point conformal block,

$$W(z_N, x_N, ..., z_1, x_1) \qquad (34)$$

is a function of N pairs of variables, (z_i, x_i). The proposal of Furlan, Ganchev, Paunov and Petkova [11] which we have alluded to in the introduction, is that when we put

$$x_i = z_i \qquad (35)$$

then this block agrees up to normalisation with a corresponding block in the minimal conformal theory with the same p, q as for the $\widehat{SL(2)}_k$ theory with $t = k + 2 = p/q$. This statement was verified in many examples in ref. [11]. Using our technique we are in a position to present a proof and also to clarify how the result is related to the

more standard version of hamiltonian reduction, based on

$$J^+(z) \sim 1 \qquad (36)$$

In addition we partly find a somewhat stronger result, partly we also find that the factor of proportionality may easily become zero. Further, the behaviour of the conformal block as $x_i \to z_i$ can be non-holomorphic. In these cases the program of [11] would seem to be in difficulty. More precisely, we have found that one has the following

$$\begin{aligned}
& W_N^{WZW}(\{z_i, x_i = x \cdot z_i\}) \\
= \; & c_N(\{j_l\}, x) W_N^{\text{minimal}}(\{z_i\}) \qquad (37)
\end{aligned}$$

and we have obtained the coefficient, c_N. The treatment is based on the observation that since

$$[J_n^a, \phi_j(z, x)] = z^n D_x^a \phi_j(z, x) \qquad (38)$$

then

$$\begin{aligned}
\phi_j(z, x \cdot z) & = e^{zxJ_0^-} \phi_j(z, 0) e^{-zxJ_0^-} \\
& = e^{xzD_y^-} \phi_j(z, y)|_{y=0} \\
& = e^{xJ_1^-} \phi_j(z, 0) e^{-xJ_1^-} \\
& = e^{xJ_1^-} : e^{-j\sqrt{\frac{2}{t}}\varphi(z)} : e^{-xJ_1^-} \qquad (39)
\end{aligned}$$

We may then express the N-point conformal block as follows

$$\begin{aligned}
W_N & = \langle j_N | \phi_{j_{N-1}}(z_{N-1}, x \cdot z_{N-1}) ... \\
& \quad ... \; \phi_{j_2}(z_2, x \cdot z_2) \prod_i \oint \frac{dw_i}{2\pi i} S_{k_i}(w_i) | j_1 \rangle \\
& = \langle j_N | e^{xJ_1^-} \phi_{j_{N-1}}(z_{N-1}, 0) ... \phi_{j_2}(z_2, 0) \\
& \quad \prod_i \oint \frac{dw_i}{2\pi i} \beta(w_i)^{-k_i} e^{-k_i \sqrt{\frac{2}{t}} \varphi(w_i)} | j_1 \rangle (40)
\end{aligned}$$

Formally this is completely straightforward, however there is a tricky mode question which must be examined when identities like

$$e^{-xJ_1^-} e^{xJ_1^-} = 1 \qquad (41)$$

are used, since it turns out that the two exponentials in general require different fractional expansions. However, all is well [22]. The crucial observation now is that it may be shown that

$$\langle j_N | e^{xJ_1^-} = \langle j_N | (1 - x\gamma_1)^{k-2j_N} (-)^{-r+st}$$

$$\cdot \; \frac{\sin\{\pi(k - 2j_N)\}}{\sin\{\pi(k - 2j_N - r + st)\}} \qquad (42)$$

where again r, s denote the number of screening operators of the first and second kind respectively. We therefore see that the only γ dependence in the correlator is via the mode γ_1. This mode only interacts with the β_{-1} mode which in turn multiplies $w_i^0 = 1$. Thus the $\gamma\beta$ part in the above correlator is trivial and all the remaining ingredients are exactly a free field representation of a minimal model correlator. This completes our proof of the FGPP proposal [11]. Thus the formulation of hamiltonian reduction at the level of conformal blocks has an extremely simple realization: one has to put the x-variables in the $SL(2)$ correlator proportional to the z variables. In the next subsection we examine how the result is related to more standard formulations of hamiltonian reduction.

5.1. Relation to standard hamiltonian reduction

In our case the standard formulation of hamiltonian reduction is in terms of imposing the constraint

$$J^+(z) = 1 \qquad (43)$$

It is convenient to view this constraint as a two step process: (i) first, all modes except J_{-1}^+ are not operative, this makes $J^+(z)$ act like a constant, and (ii) the constant is put equal to a definite value, like 1. In our free field formulation, $J^+(z)$ is represented by $\beta(z)$. We see that in order for a correlator to respect the constraint, only the mode $J_{-1}^+ = \beta_{-1}$ must be active. This in turn requires that the only γ dependence that can be tolerated is via γ_1 and hence via J_1^-. We see that this is precisely the crucial property we observed above. Thus the most general (chiral) correlator respecting the constraint will have to be of the form

$$\begin{aligned}
W_{\text{constr}} & = \langle j_N | F(J_1^-) \phi_{j_{N-1}}(z_{N-1}, 0) ... \\
& \quad ... \; \phi_{j_2}(z_2, 0) \prod_i \oint \frac{dw_i}{2\pi i} \beta(w_i)^{-k_i} \\
& \quad \cdot \; e^{-k_i \sqrt{\frac{2}{t}} \varphi(w_i)} | j_1 \rangle
\end{aligned}$$

$$= \langle j_N | f(\gamma_1) \phi_{j_{N-1}}(z_{N-1}, 0) ...$$

$$... \; \phi_{j_2}(z_2, 0) \prod_i \oint \frac{dw_i}{2\pi i} \beta(w_i)^{-k_i}$$

$$\cdot \; e^{-k_i \sqrt{\frac{2}{t}} \varphi(w_i)} | j_1 \rangle \tag{44}$$

We have shown that our $SL(2)$ correlator with $x_i = x \cdot z_i$ is exactly of this form. In order for the value of the constant to be 1 it is required that

$$f(\gamma_1) = e^{-\gamma_1} \tag{45}$$

This condition is not satisfied by our correlator, hence there is a non trivial constant of proportionality between the reduced $SL(2)$ correlator and the minimal model one.

6. OUTLOOK

It appears that we have solved a rather non-trivial problem, namely that of writing down $SL(2)$ conformal blocks for admissible representations. Equivalently our technique contains a powerful method for generating solutions of the Knizhnik-Zamolodchikov equations. We emphasise that all this is achieved with the physically convenient free field technique, although it was required to apply fractional calculus to do that.

Several more steps have to be taken before we can achieve our original goal of treating 2-d quantum gravity with this technique and in particular generalise that. Here we mention some of these steps most of which represent work in progress.

We must understand the normalization of the three point functions in order to calculate 2-d gravity dressed ones using the $SL(2)/SL(2)$ cohomology. This requires us to extend the program of [29] to the present case. On this problem we are currently working. Next we should generalise the free field formalism to higher groups and supergroups. Several steps in this direction have already appeared in the literature, but we need to extend that. Here we have some partial results. It would be interesting to understand the meaning of the constraint $x_i = z_i$ on higher genus Riemann surfaces and indeed for higher groups (W-like strings). These and more questions need be examined before we can truly use G/G theories as a general tool to study generalised non-critical string theory.

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